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Research Note

The expressive power of circumscription

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Abstract

Circumscription is a form of nonmonotonic reasoning, introduced by McCarthy (1997) as a way of characterizing defaults using second order logic. The consequences of circumscription are those formulas true in the minimal models under a pre-order on models. In the case of domain circumscription the pre-order was the sub-model relation. Formula circumscription (McCarthy, 1980, 1986) is characterized by minimizing a set of formulas—one model is preferred to another model when the extensions of the minimized formulas in the first are subsets of the extensions in the second.

We show that the propositional version of formula circumscription can capture all pre-orders on valuations of finite languages. We consider the question of infinite languages, and give the corresponding representation theorems. We further show that there are natural defaults (inertia in temporal projection), captured by inductive definitions, that cannot be captured by circumscription in the first order case.

Finally, contrary to previous claims, we show that propositional formula circumscription can capture all preferential consequence relations over finite propositional languages, as defined by Kraus et al. (1990). Thus, in the finite propositional case, there is no restriction on the kinds of preferential defaults that circumscription can describe. © 1998 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

McCarthy's circumscription [20] is the first application of the idea of relative likelihood, or preference between models, to nonmonotonic reasoning. There, he recognized that a natural way of representing defaults was to order states of the world, according to what we thought was the case, and then to choose the sentences true in the minimal models in

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this order as our current beliefs. This has strong links to the notion of inductive definition and to the use of minimality as a definitional tool which is ever-present in mathematics. This basic idea has since been studied by many researchers. The original suggestion was to order models based on the size of their domain, preferring sub-models, analogously to mathematical induction. This was later changed to minimizing the extension of a predicate, and then of an arbitrary formula.

The notion of minimal entailment has been considered by many researchers. It has one of its roots in Ramsey's [25] definition of counterfactuals:

> (Ramsey Test) Accept a propositional of the form "if A then C" in a state of belief K if and only if the minimal change of K needed to accept A also requires accepting C.

Since then the idea of representing counterfactuals using entailment in minimal models has been widely considered [14,28]. Gärdenfors [8,9], and Makinson [11] have considered similar themes in their work on belief revision, which has close connections with nonmonotonic reasoning [12]. Shoham [27] was the first to examine the model theory of McCarthy's idea of using minimal entailment for nonmonotonic reasoning. Since then, this notion of preferential entailment has become linked with approaches [7] based on characterizing nonmonotonicity through structural rules.

Despite the large amount of interest the area has attracted, some of the basic questions about circumscription remain unanswered. What is circumscription's expressive power? How do the various forms of circumscription proposed compare in expressive power? How should the expressive power of nonmonotonic systems be measured?

In the next sections we attempt to answer these questions. We first consider the structural meaning of minimizing predicates or formulas. We find that minimizing a formula is equivalent to stating that the models that satisfy that formula are an upper-set of the pre-order on models. Using this idea, it immediately follows that all pre-orders are definable using circumscription in the finite propositional case.

As the expressive power of first order logic with a fixed domain and propositional logic co-incide, this shows that circumscription captures all pre-orders on structures, for first order theories with a fixed domain.

When we look at infinite languages, we see that the circumscription formula can only describe pre-orders that are the product of a finite and a discrete pre-order.

Infinite languages, and infinite sets of minimized formulas can still not capture all pre-orders—only those pre-orders whose upper-sets have a basis, all of whose elements are finitely axiomatizable. Even if we minimize a countable set of infinitary formulas, we still cannot capture all pre-orders.

We show that a generalization of circumscription, suggested by Lifschitz [15] is strictly more expressive than formula circumscription in the first order case. We show a default commonly used in reasoning about action cannot be captured by formula circumscription, but can be captured by Lifschitz's generalized circumscription.

Finally we look at preferential consequence relations as defined by Kraus et al. [13]. We show that formula circumscription of propositional theories can capture all preferential consequence relations over finite propositional languages. Previously Kraus et al. [13, p. 169] have claimed that circumscription was unable to characterize these relations.

Circumscription was among the first systems for nonmonotonic reasoning that was proposed. In recent years, it has been criticized for lack of expressive power, especially in comparison to approaches based on consequence relations. In this paper, we show that these criticisms were unfounded.

2. Notation

In this section we recap some notation we need concerning orders.

Definition 1 (*Pre-order*). A pre-order, $\leqslant$, on a domain D, is a binary relation that is transitive and reflexive; we denote a pre-order on D, as $\leqslant_D$, we drop the subscript when the domain is obvious:

$$\forall d, d', d'' \in D.d \leqslant d' \text{ and } d' \leqslant d'' \text{ imply } d \leqslant d'' \text{ (transitivity)},$$

$$\forall d \in D.d \leqslant d \text{ (reflexivity)}.$$

Definition 2 (*Partial order*). A partial order on a domain D is a pre-order that is anti-symmetric; we denote a partial order on D, as $\leqq_D$, we drop the subscript when the domain is obvious:

$$\forall d, d' \in D.d' \leqq d \text{ and } d \leqq d' \text{ imply } d = d' \text{ (anti-symmetry)}.$$

The discrete partial order on D is the identity on D. We say a pre-order is finite when its domain is finite.

Definition 3 (*Upper-set*). An upper-set $\mathcal{F}$ of a pre-order $\leqslant$ on a domain D is a subset of D that is upwards closed, that is

$$\forall d \in \mathcal{F}, d' \in D.d \leqslant d' \text{ implies } d' \in \mathcal{F}.$$

Upper-sets are sometimes called order-filters.

Proposition 1. *The family of all upper-sets of a pre-order $\leqslant_D$ is closed under arbitrary unions and intersections.*

Proposition 2. *Every set of sets X is closed under arbitrary unions and intersections is the family of upper-sets of a unique pre-order $\leqslant$.*

Definition 4 (*Basis*). A basis B for F, a family of upper-sets of a pre-order $\leqslant$, is a subset of F, such that every element of F is definable in terms of the intersection and union of elements of B:

$$\forall f \in F.\exists b_{1,1}, \ldots, b_{n,m} \in B.\bigcup_{i=1}^{n}\bigcap_{j=1}^{m} b_{i,j} = f.$$

Definition 5 (*Minimal*). An element d is said to be a minimal element in an order $\leqslant$, on a domain D, if there is no element $d' \in D$ such that $d' \leqslant d$ and $d \nleqslant d'$.

$$Minimal_{\leqslant}(d) \equiv d \in D \wedge \forall d' \in D.d' \leqslant d \rightarrow d \leqslant d'.$$

Definition 6 (*Restriction*). The restriction of a pre-order $\leqslant_D$, to a subset D' of D, is written, $\leqslant_{\downarrow D'}$.

3. The model theory of circumscription

In this section we look at a commonly used form of circumscription. This version includes the modifications introduced by Lifschitz [16] in his reformulation of formula circumscription [22]. It also uses the device, introduced by de Kleer and Konolige [5], of fixing predicates by minimizing both the predicate and its negation. We define formula circumscription and its semantics in the propositional case. We then look at some of the properties of the pre-orders defined by circumscription. We see that they are not always partial orders, and that in the case of finite languages, minimal valuations always exist.

We then see that the formulas true in the minimal models of this pre-order on models are exactly the consequences of circumscription. We then consider the question of what pre-orders circumscription can capture. As our first step towards this we look at how minimized formulas are related to the pre-order on models they generate. We see that the set of models of each minimized formula is an upper-set of the pre-order on models. We next show that the pre-order is the unique pre-order that has these upper-sets as a basis. This establishes the basic representation result for circumscription—every pre-order of the valuations of a finite propositional language can be captured by a circumscription policy.

We then show the restriction to finite languages is necessary.

3.1. Propositional formula circumscription

McCarthy [22] introduced a form of circumscription, called *formula circumscription*. It extends predicate circumscription [21] in two ways.

- The new form of circumscription allows the extensions of arbitrary formulas to be minimized rather than only minimizing predicates.
- Certain predicates are allowed to be *varied* in the minimization process. This notion was suggested, but not explored, by [21].

Definition 7 (*$\leqslant$ on formulas*). It is useful to introduce the notation $\gamma_1, \gamma_2 \leqslant \delta_1, \delta_2$ for the formula $(\gamma_1 \rightarrow \delta_1) \wedge (\gamma_2 \rightarrow \delta_2)$.

Definition 8 (*McCarthy* [22]). If γ_1, γ_2 is a pair [2] of formulas from a propositional language L, and $P_1, \ldots, P_n$ are the propositional letters in our language L, then we define the circumscription of α, a formula in L minimizing γ_1, γ_2, written $Circ(\alpha; \gamma_1, \gamma_2)$ as follows.

[2] For notational simplicity we minimize two formulas, the extension to more formulas is the natural one.

We use α' to denote the formula, $\alpha[\psi_1/P_1, \ldots, \psi_n/P_n]$, that is the result of the replacement of the propositional letters P_i in α by the propositional variables ψ_i.

$$\alpha \wedge \forall \psi_1, \ldots, \psi_n . \left[\alpha' \wedge \gamma_1', \gamma_2' \leqslant \gamma_1, \gamma_2\right] \rightarrow \gamma_1, \gamma_2 \leqslant \gamma_1', \gamma_2'.$$

We use the notation $Circ_{L_{\overline{P}}}$ when we wish to stress that the propositional letters in the language are the set $\overline{P}$. This may differ from the letters that occur in α.

3.2. Semantics of propositional circumscription

Definition 9 ($\leqslant^{\Gamma}$). Given a propositional language $L_{\overline{P}}$ and a set of formulas Γ from $L_{\overline{P}}$, we define the relation $\leqslant^{\Gamma}$ on the valuations of $L_{\overline{P}}$ as follows.

We denote the truth value of a formula γ in a valuation $\mathfrak{A}$ as $\mathfrak{A}[\gamma]$. For any two valuations $\mathfrak{A}$, $\mathfrak{B}$, we write $\mathfrak{A} \leqslant^{\Gamma} \mathfrak{B}$ if

$$\mathfrak{A}[\gamma] = 1 \text{ implies } \mathfrak{B}[\gamma] = 1 \text{ for every } \gamma \text{ in } \Gamma.$$

We now look at some of the properties of $\leqslant$. It is a pre-order on the class of all valuations of a given propositional language. It is not in general a partial order as there are distinct valuations that are $\leqslant^{\Gamma}$ than each other.

Proposition 3. *For a given propositional language $L_{\overline{P}}$, and a set of formulas Γ from $L_{\overline{P}}$, $\leqslant^{\Gamma}$ is a pre-order but not always a partial order.*

Proof. That $\leqslant^{\Gamma}$ is a pre-order follows from the transitivity and reflexivity of implication.

To show transitivity we assume that

$$\mathfrak{A} \leqslant \mathfrak{B} \quad \text{and} \quad \mathfrak{B} \leqslant \mathfrak{C}$$

and show that this implies that $\mathfrak{A} \leqslant \mathfrak{C}$. To show that $\mathfrak{A} \leqslant \mathfrak{C}$, it suffices to show that for each $\gamma \in \Gamma$, if $\mathfrak{A}[\gamma] = 1$, then $\mathfrak{C}[\gamma] = 1$. Thus we assume that $\mathfrak{A}[\gamma] = 1$ and show this implies $\mathfrak{C}[\gamma] = 1$.

As $\mathfrak{A} \leqslant \mathfrak{B}$, then

$$\mathfrak{A}[\gamma] = 1 \text{ implies } \mathfrak{B}[\gamma] = 1.$$

Therefore, $\mathfrak{B}[\gamma] = 1$. But, also, as $\mathfrak{B} \leqslant \mathfrak{C}$, then $\mathfrak{C}[\gamma] = 1$, as required.

To show reflexivity, it is sufficient to show that if $\mathfrak{A}[\gamma] = 1$ then $\mathfrak{A}[\gamma] = 1$, which is immediate.

To see that it is not always a partial order, let Γ be $\{P\}$, the language be $L_{\{P,Q\}}$. Then the two valuations $\mathfrak{A}$, and $\mathfrak{B}$, where

$$\mathfrak{A}[P] = 0, \quad \mathfrak{A}[Q] = 0, \quad \mathfrak{B}[P] = 0, \quad \mathfrak{B}[Q] = 1,$$

provide a counterexample to anti-symmetry. □

As we considered earlier there are minimal elements in many pre-orders. Thus we can define the minimal models of a theory with respect to $\leqslant^{\Gamma}$.

Proposition 4. *Let α be a formula of $L_{\overline{P}}$, and Γ a finite set of formulas of $L_{\overline{P}}$. A valuation $\mathfrak{A}$ for a finite set of propositional letters $\overline{P}$ is a model of $Circ_{L_{\overline{P}}}(\alpha; \Gamma)$, if and only if $\mathfrak{A}$ is minimal in the set of models of α with respect to $\leqslant^{\Gamma}$.*

Proof. We show the two propositions are inter-derivable. We first note that both propositions imply that $\mathfrak{A}$ is a model of α. We then show the second conjuncts of both are inter-derivable.

$$
\begin{aligned}
&Minimal_{\leqslant^{\Gamma}\downarrow\{\mathfrak{C}\mid\mathfrak{C}\models\alpha\}}\mathfrak{A}\\
&\quad\equiv \forall\mathfrak{B}.\mathfrak{B}\models\alpha\wedge(\mathfrak{B}\leqslant^{\Gamma}\mathfrak{A})\rightarrow(\mathfrak{A}\leqslant^{\Gamma}\mathfrak{B})\\
&\quad\equiv \forall\mathfrak{B}.\mathfrak{B}\models\alpha\wedge(\forall\gamma\in\Gamma.\mathfrak{B}[\gamma]\rightarrow\mathfrak{A}[\gamma])\rightarrow(\forall\gamma\in\Gamma.\mathfrak{A}[\gamma]\rightarrow\mathfrak{B}[\gamma])\\
&\quad\equiv \forall\psi_1,\ldots,\psi_n.\alpha[\psi_1/P_1,\ldots,\psi_n/P_n]\wedge\\
&\qquad \gamma_1[\psi_1/P_1,\ldots,\psi_n/P_n],\ldots,\gamma_n[\psi_1/P_1,\ldots,\psi_n/P_n]\leqslant\gamma_1,\ldots,\gamma_n\rightarrow\\
&\qquad\quad \gamma_1,\ldots,\gamma_n\leqslant\gamma_1[\psi_1/P_1,\ldots,\psi_n/P_n],\ldots,\gamma_n[\psi_1/P_1,\ldots,\psi_n/P_n].
\end{aligned}
$$

The first equivalence follows from the definition of *Minimal*. The second follows from the definition of $\leqslant^{\Gamma}$. The third equivalence is valid as a model can be viewed as a valuation function, and from the definition of satisfiability. Note this requires that there be only a finite number of propositional letters. The last formula is the second conjunct in the definition of circumscription. $\square$

Thus we have shown that minimal entailment over pre-orders on valuations describe circumscription. The next obvious question is what pre-orders can circumscription capture?

3.3. Characterization of minimized formulas

We begin by relating the minimized formulas to properties of the pre-order,

Proposition 5. *Given a set Γ of formulas from a propositional language $L_{\overline{P}}$, the set of models of each $\gamma \in \Gamma$ is an upper-set of the pre-order $\leqslant^{\Gamma}$.*

Proof. We need to show that the models of γ are upward closed. Let Γ be a set of formulas of a propositional language. Let γ be an element of Γ. Let $\mathfrak{A}$ be a model of γ, and let $\mathfrak{B}$ be a valuation such that $\mathfrak{A}\leqslant^{\Gamma}\mathfrak{B}$. We show that $\mathfrak{B}\models\gamma$. As $\mathfrak{A}\leqslant^{\Gamma}\mathfrak{B}$ we have $\mathfrak{A}[\gamma]=1$ implies $\mathfrak{B}[\gamma]=1$ for every γ in Γ. We have $\mathfrak{A}\models\gamma$, by assumption. Therefore $\mathfrak{B}\models\gamma$ as required. $\square$

This gives a structural interpretation of minimizing a formula. The set of models of a minimized formula is an upper-set of the pre-order on valuations.

We now show that the pre-order $\leqslant^{\Gamma}$ is the pre-order with the fewest upper-sets that contains all the upper-sets in Γ.

Proposition 6. *Given a set Γ of formulas of a propositional language $L_{\overline{P}}$ the pre-order $\leqslant^{\Gamma}$ is the unique pre-order whose upper-sets include the models of each formula in Γ and whose set of upper-sets is minimal.*

Proof. We know that every formula in Γ generates an upper-set of $\leqslant^{\Gamma}$. We further know that arbitrary unions and intersections of upper-sets are upper-sets by Proposition 1. We consider the set of sets of models M, whose elements are the set of models of each element of Γ. We close out M under arbitrary unions and intersections, to get M'. M' is a set of upper-sets, as M' is closed under unions and intersections, by Proposition 2. M' is minimal as any smaller set would deny that every set in M was an upper-set. As every set of sets closed under unions and intersection corresponds to a unique pre-order, M' corresponds to a unique pre-order. M' includes the models of each formula in Γ by construction.

Thus we need to show that this M' is the set of upper-sets of $\leqslant_{\Gamma}$. We show every element of M' is an upper-set of $\leqslant_{\Gamma}$. Every element of M' is definable in terms of union and intersection from element of M, which are upper-sets of Γ. Thus, by Proposition 1, every element of M' is an upper-set of $\leqslant_{\Gamma}$.

Finally we show that every upper-set of $\leqslant_{\Gamma}$ is an element of M'. We consider an arbitrary upper-set F of $\leqslant_{\Gamma}$. We can divide F into disjoint sets F_{Δ}, each of which satisfy a different subset Δ of Γ. If there is a model $\mathfrak{A} \in F$ such that for $\gamma \in \Gamma$, $\mathfrak{A} \models \gamma$ if and only if $\gamma \in \Delta$, then every $\mathfrak{B}$, such that $\gamma \in \Gamma$, $\mathfrak{B} \models \gamma$ if $\gamma \in \Delta$, is in F, as F is upwards closed, and $\mathfrak{B} \leqslant_{\Gamma} \mathfrak{A}$, by the definition of $\leqslant_{\Gamma}$. The set of all $\mathfrak{B}$'s such that $\mathfrak{B} \models \gamma$ if $\gamma \in \Delta$ is the intersection of the sets, $\{\mathfrak{C} \mid \mathfrak{C} \models \delta\}$. Each of these sets $\{\mathfrak{C} \mid \mathfrak{C} \models \delta\}$ is an upper-set.

Thus if F contains $\mathfrak{A}$, such that for $\gamma \in \Gamma$, $\mathfrak{A} \models \gamma$ if and only if $\gamma \in \Delta$, then F contains the intersections of the upper-sets $\{\mathfrak{C} \mid \mathfrak{C} \models \delta\}$ for each $\delta \in \Delta$. We call this intersection, F'_{Δ}. We will need the fact that $F_{\Delta} \supset F'_{\Delta}$, which follows from the definition of F'_{Δ}.

Further, as F is the union of the F_{Δ}'s, and each F_{Δ} is contained in the respective F'_{Δ}'s, which are in turn contained in F, then F is the union of the F'_{Δ}'s. Thus F is the union of intersections of upper-sets from M, as every $\{\mathfrak{C} \mid \mathfrak{C} \models \delta\}$ for $\delta \in \Delta$ is an element of M, as $\Delta \subset \Gamma$. Thus every upper-set of $\leqslant_{\Gamma}$ is an element of M'. □

We now use the fact that every pre-order is uniquely defined by its set of upper-sets, to show that every pre-order on the valuations of a finite propositional language is defined by some $\leqslant^{\Gamma}$.

Theorem 7. *If $\leqslant$ is a pre-order on the valuations of a finite propositional language $L_{\overline{P}}$, then there is a finite set of formulas Γ, such that $\leqslant = \leqslant^{\Gamma}$.*

Proof. Let Γ' be the set of all formulas whose models are upper-sets of $\leqslant$. We know there are only a finite number of logically inequivalent formulas as, by assumption, our language is finite. We now form a set Γ by choosing one formula from each set of logically equivalent formulas in Γ'. This set contains all formulas whose set of models are upper-sets, and is finite, thus it generates the unique pre-order which has exactly those upper-sets, by Proposition 6. □

This shows that circumscription can capture all pre-orders on finite propositional languages.

3.4. First order finite domains

In Artificial Intelligence the case where we need only consider theories with domains smaller than a fixed size is often useful. For these structures the results for propositional languages with finite number of letters apply, by considering the well-known translation of fixed finite domain theories into propositional logic.

Thus for fixed finite domain theories, formula circumscription can capture all pre-orders on structures. For the case where our theories have infinite models results are more difficult to come by. We no longer have categoricity, that is, we cannot specify a model up to isomorphism by a formula, or even a set of formulas. In our previous construction 7, we explicitly use the fact that every upper-set of the pre-order on models is axiomatized by a single formula.

3.5. Infinitary cases

Note that the restriction to finite propositional languages was necessary in the previous theorem. We give an example.

Example 1. Let L be the propositional language containing the propositional letters $P_1, \ldots, P_n, \ldots,\ n \in \omega$. Consider the following pre-order $\leqslant_1$ on L-valuations. Let one equivalence class S be the set of valuations where exactly one propositional letter is true. Let the other class $\overline{S}$ be the complement of this class. Let all elements of S be greater than every element of $\overline{S}$.

There is no formula γ that is satisfied only in S. Let us suppose that there was. Choose a valuation that agrees with a valuation in S on all letters that are mentioned in the formula γ, but which is not in S. This valuation satisfies γ, and is not in γ, thus there is no formula γ that is satisfied only in S.

Thus for this pre-order, the set of non-empty, non-trivial upper-sets that are represented by formulas is empty. However, the empty set of formulas is also a basis for the discrete pre-order, the identity on the domain.

Thus, there is no set of formulas Γ, such that $\leqslant_\Gamma = \leqslant_1$. Any possible set of formulas Γ must contain only formulas whose sets of models are upper-sets of $\leqslant_1$, by Proposition 5. But the only formulas that are upper-sets of $\leqslant_1$ are tautologies and the negations of tautologies. Thus, Γ must be a set of tautologies and the negations of tautologies. But then $\leqslant_\Gamma$ is the discrete pre-order, not $\leqslant_1$ which shows that there is no set of formulas Γ, such that $\leqslant_\Gamma = \leqslant_1$.

We can achieve an analogue of Theorem 7 for the infinite case, if we allow Γ to be infinite. For this analogue we need to add a restriction that certain upper-sets are finitely axiomatizable. This is necessary as shown above.

Theorem 8. *If $\leqslant$ is a pre-order of the valuations of a propositional language $L_{\overline{P}}$ then there is a set of formulas Γ, such that $\leqslant = \leqslant^\Gamma$ if and only if there is a basis for the upper-sets of $\leqslant$ where each element in the basis is finitely axiomatizable.*

Proof. ($\Leftarrow$) We take the axiomatizations of the finitely axiomatizable upper-sets as our set Γ. By Proposition 6 the pre-order $\leqslant^{\Gamma}$ is the unique pre-order whose upper-sets include the models of each formula in Γ and whose set of upper-sets is minimal. Thus given a set of formulas Γ, that axiomatize a basis for a set of upper-sets, $\leqslant_{\Gamma}$ generates the pre-order with that set of upper-sets. As there is a finitely axiomatizable basis for the upper-sets of $\leqslant$, the finitely axiomatizable upper-sets of $\leqslant$ (which are axiomatized by Γ) form a basis, and thus $\leqslant_{\Gamma} = \leqslant$.

($\Rightarrow$) We prove the contrapositive.

Suppose there is no basis, all of whose members are finitely axiomatizable. Then consider all finitely axiomatizable upper-sets. Let these be Γ'. Then our assumption is that there is an upper-set $\mathcal{F}$, that is not definable in terms of the elements of unions and intersection of elements of Γ'.

Every pre-order $\leqslant^{\Gamma}$ with $\Gamma \subset \Gamma'$ does not contain $\mathcal{F}$, and thus is not equal to $\leqslant$. Every pre-order $\leqslant^{\Gamma}$ with $\Gamma \supsetneq \Gamma'$ contains an upper-set axiomatized by some γ that is not in Γ'. Therefore, $\leqslant^{\Gamma}$ has some new γ, whose set of models is an upper-set. However, this new γ is finitely axiomatizable, and thus its set of models is not an upper-set of $\leqslant$, as all finitely axiomatizable upper-sets are elements of Γ', by assumption. Thus $\leqslant^{\Gamma}$ contains an extra upper-set and is not equal to $\leqslant$, as required. $\square$

This gives us a characterization of what minimizing infinite sets of formulas can capture. It is notable that the condition that there is a basis which is finitely axiomatizable is a weaker condition that every upper-set being finitely axiomatizable.

3.6. Finite strings of quantifiers

In the above theorem we allow Γ to be infinite. If we wish to capture exactly what a circumscription formula can represent, we need to look at the case where Γ is finite, and where the number of universally quantified propositional variables in the circumscription formula is finite.

We now give a representation theorem for circumscription. We characterize exactly those pre-orders that propositional formula circumscription can define. They are the pre-orders that can be factored into a finite pre-order and a discrete, or unconnected, pre-order. We first make the notion of the product of pre-orders precise and then state the result.

Definition 10. Given a pre-order $\leqslant_D$ and a pre-order $\leqslant_{D'}$ the pre-order $\leqslant_{D\times D'} = \leqslant_D \times \leqslant_{D'}$ is defined over all pairs $\langle d, d'\rangle$, $d \in D$, $d' \in D'$, such that:

$$\forall d_1, d_2, d_1', d_2'.\ d_1 \leqslant_D d_2 \wedge d_1' \leqslant_{D'} d_2' \equiv \langle d_1, d_1'\rangle \leqslant_{D\times D'} \langle d_2, d_2'\rangle.$$

Lemma 9. *If $\leqslant$ is a pre-order on valuations of a language $L_{\overline{P}}$, and $\leqslant'$ is a pre-order on valuations of a language $L_{\overline{P'}}$ where $\overline{P}$ and $\overline{P'}$ are disjoint then $\leqslant \times \leqslant'$ is isomorphic to a pre-order on valuations of a language $L_{\overline{P}\cup\overline{P'}}$.*

Proof. We consider the pair of valuations to be a single valuation over the union of the propositional letters. This is then a pre-order on valuations of $L_{\overline{P}\cup\overline{P'}}$. $\square$

We now define a pre-order $\leqslant^{\Gamma;\overline{P'}}$ that betters captures the notion that the number of universally quantified variables in the circumscription formula is finite. $\overline{P'}$ is a finite subset of the letters in the propositional language and corresponds to the letters replaced by variables in the circumscription formula. These can alternately be considered to the the letters that are *varied* (in the sense of [22]) in the minimization process.

Definition 11 ($\leqslant^{\Gamma;\overline{P'}}$). Given a propositional language $L_{\overline{P}}$ and a finite set of formulas Γ in $L_{\overline{P}}$ and a finite set $\overline{P'} \subset \overline{P}$, we define the relation $\leqslant^{\Gamma;\overline{P'}}$ on the valuations of $L_{\overline{P}}$ as follows. For any two valuations $\mathfrak{A}$, $\mathfrak{B}$ on $\overline{P}$, we write $\mathfrak{A} \leqslant^{\Gamma;\overline{P'}} \mathfrak{B}$ if

- $\mathfrak{A}[\gamma] = 1$ implies $\mathfrak{B}[\gamma] = 1$ for every γ in Γ.
- $\mathfrak{A}[P_i] = \mathfrak{B}[P_i]$ for every P_i such that $P_i \in \overline{P}$ and $P_i \notin \overline{P'}$.

Here we give the representation theorem for propositional formula circumscription over propositional languages with infinite sets of propositional letters.

Theorem 10. *If $\leqslant$ is a pre-order on the valuations of a propositional language $L_{\overline{P}}$ then there is a set of formulas Γ from $L_{\overline{P'}}$ and a finite subset $\overline{P'}$ of $\overline{P}$, such that $\leqslant\ =\ \leqslant^{\Gamma;\overline{P'}}$ if and only if there are pre-orders $\leqslant_f$ and $\leqslant_d$ on valuations of disjoint subsets of $\overline{P}$ where $\leqslant_f$ is finite and $\leqslant_d$ is discrete and $\leqslant\ \cong\ (\leqslant_f \times \leqslant_d)$.*

Proof. ($\Rightarrow$) Let the two disjoint subsets of $\overline{P}$ be $\overline{P'}$ and $\overline{P} - \overline{P'}$. Let valuations on $\overline{P'}$ be ordered by $\leqslant^{\Gamma}$, this will be $\leqslant_f$. It is finite, as $\overline{P'}$ is finite. Let $\leqslant_d$ be the identity on valuations on $\overline{P} - \overline{P'}$. This is a discrete order.

We now need to show that the product of these pre-orders is the required $\leqslant$. We consider two valuations $\mathfrak{A}$ and $\mathfrak{B}$ ordered by $\leqslant$. We show that they are ordered by $\leqslant_f \times \leqslant_d$. By definition if $\mathfrak{A}$ and $\mathfrak{B}$ are ordered, then they agree on all letters in $\overline{P} - \overline{P'}$. Thus, the restrictions of $\mathfrak{A}$ and $\mathfrak{B}$ to the letters in $\overline{P} - \overline{P'}$ are ordered by $\leqslant_d$. Similarly if $\mathfrak{A}$ and $\mathfrak{B}$ are ordered, then by the second clause in the definition of $\leqslant$, their restriction to $\overline{P'}$ are ordered by $\leqslant_f$.

If $\mathfrak{A}$ and $\mathfrak{B}$ not ordered by $\leqslant$, a similar case analysis show that they either disagree on the valuations of the letters in $\overline{P} - \overline{P'}$, and are not ordered by $\leqslant_d$, or their restrictions to $\overline{P'}$ are not ordered by $\leqslant_f$. Therefore, $\leqslant_d \times \leqslant_f\ \cong\ \leqslant$, as required.

($\Leftarrow$) We are given $\leqslant_f$ and $\leqslant_d$. Take Γ to be a finite set of formulas such that $\leqslant_f\ =\ \leqslant^{\Gamma}$. This exists by our earlier theorem. Let $\overline{P'}$, be the disjoint subset of $\overline{P}$ that is ordered by $\leqslant_f$.

We now need to show that $\leqslant^{\Gamma;\overline{P'}}\ =\ \leqslant$. This follows from Lemma 9, and the fact that $\leqslant_f\ =\ \leqslant^{\Gamma}$, and that $\leqslant^{\Gamma;\overline{P'}}$ restricted to letters outside $\overline{P}$ is the identity on valuations. $\square$

Thus we have shown that our propositional formula circumscription captures all pre-orders of a certain general class. Pre-orders over models have been proposed as a general semantic framework for nonmonotonic reasoning. Thus we have shown that circumscription captures this notion of nonmonotonic reasoning.

3.7. *Infinite strings of quantifiers*

In the last two theorems we saw that circumscription's ability to define pre-orders was impaired by the finite nature of the the circumscription formula. It is possible to consider languages with infinitary connectives. Chapter 2 of [2] considers the propositional languages L_{ω_1}, which allow countable conjunctions and disjunctions of formulas. These languages shed light on the treatment of the first order case. A countable conjunction can be viewed as a universal quantifier over a fixed domain. As circumscription only compares models which have the same domain, this notion seems appropriate.

A direct investigation of the first order case seems difficult at this time, because of the difficulties that circumscription has in characterizing defaults about equality or the domain [6]. Neither domain closure, nor the unique names assumption are naturally captured in formula circumscription. Modifications to circumscription [17,26] that do capture these defaults have been proposed, but are significantly more complicated to analyze.

Comparison to other approaches is also made difficult by the lack of a compelling treatment of the first order case in nonmonotonic consequence relations. Circumscription has a well developed treatment of the first order case of nonmonotonic reasoning. Current proposals for first order reasoning with consequence relations are much less developed, and are highly non-trivial [18]. In particular many do not accept a Barcan like rule—rather there are competing proposals, some of which order domain objects [1], while other order structures [4].

The disadvantages of infinitary languages of this kind are that they do not have a complete proof system in the absence of infinitary proof rules.

Not every pre-order is captured by a circumscription policy even if we allow infinitary logics and restrict ourselves to countable languages.

Theorem 11 ([2, Theorem 15]). *There exists pre-orders $\leqslant$ on valuations of a countable propositional language $L_{\overline{P}}$, that are not equal to any pre-order of the form $\leqslant^{\Gamma}$, where Γ is a countable set of formulas from $L_{\omega_1,\overline{P}}$.*

4. Formula circumscription

We now consider the first order case of circumscription. That is, we consider how we circumscribe first order theories. We do not treat functions and constants in the following as the generalization sheds no new light on our exposition. Thus our signatures will not contain any function or constant symbols.

Definition 12 ($\leqslant$, $<$ *on first order formulas*). We introduce the notation $\gamma_1, \gamma_2 \leqslant \delta_1, \delta_2$ for the formula $(\forall \overline{x}.\gamma_1(\overline{x}) \to \delta_1(\overline{x})) \wedge (\forall \overline{y}.\gamma_2(\overline{y}) \to \delta_2(\overline{y}))$, and $\gamma_1, \gamma_2 < \delta_1, \delta_2$ for the formula

$$\gamma_1, \gamma_2 \leqslant \delta_1, \delta_2 \wedge \neg(\delta_1, \delta_2 \leqslant \gamma_1, \gamma_2),$$

where $\overline{x}$ and $\overline{y}$ are sequences of variables equal in number to the free variables of γ_1, δ_1 and γ_2, δ_2.

Definition 13. If γ_1, γ_2 is a pair of[3] formulas of a first order language L, and P_1 is a predicate symbol[4] of L, then the circumscription of α minimizing γ_1, γ_2, formulas in L, varying P_1, written $Circ_F(\alpha; \gamma_1, \gamma_2; P_1)$ is,

$$\alpha \wedge \forall \psi_1 . \neg\left[\alpha' \wedge \gamma_1', \gamma_2' < \gamma_1, \gamma_2\right],$$

where ψ_1 is a predicate variable equal in arity to P_1. We use α' to denote the formula, $\alpha[\psi_1/P_1]$, that is the replacement of the predicate P_1 in α by a predicate variable ψ_1.

We now introduce the semantics for formula circumscription.

Definition 14. Let α be a sentence of L, a first order language with equality, whose predicates are $P_1, \ldots, P_n$. Let Z be a subset of $P_1, \ldots, P_n$. A model $\mathfrak{A}$ of α is a $\gamma_1, \ldots, \gamma_n; Z$*-sub-model* of a model $\mathfrak{B}$, written $\mathfrak{A} \leqslant^{\gamma_1, \ldots, \gamma_n; Z} \mathfrak{B}$, if and only if:
- $\|\mathfrak{A}\| = \|\mathfrak{B}\|$, where $\|\mathfrak{A}\|$ denotes the universe of $\mathfrak{A}$,
- $\mathfrak{A}[\gamma_i] \subseteq \mathfrak{B}[\gamma_i]$, for $i = 1, \ldots, n$, where $\mathfrak{A}[\gamma_i]$ is the interpretation of γ_i in $\mathfrak{A}$,
- $\mathfrak{A}[K] = \mathfrak{B}[K]$, for every predicate constant not in Z.

$\mathfrak{A}$ is a $\gamma_1, \ldots, \gamma_n; Z$-minimal model of α if and only if it is a minimal model of α under $\leqslant^{\gamma_1, \ldots, \gamma_n; Z}$.

Thus, as in the propositional case, every circumscription policy is a finite set of formulas, and it gives rise to a pre-order.

We now consider another way of defining pre-orders on L-structures. We first need the notion of a disjoint union of structures.

Definition 15. Let L be a first order language, with predicates $P_1, \ldots, P_n$. If $\mathfrak{M}$ and $\mathfrak{N}$ are L structures with the same universe, then we define the disjoint union of $\mathfrak{M}$ and $\mathfrak{N}$ to be a first order structure $\mathfrak{O}$ over a language $P_1, \ldots, P_n, P_1', \ldots, P_n'$, such that the interpretation of P_i in $\mathfrak{M}$ is the interpretation of P_i in $\mathfrak{O}$ and the interpretation of P_i in $\mathfrak{N}$ is the interpretation of P_i' in $\mathfrak{O}$.

We now define what a *first order definable* pre-order on L-structures is.

Definition 16. Let L be a first order language, with predicates $P_1, \ldots, P_n$, and $\leqslant$ a pre-order on structures of L. We say that $\leqslant$ is *first order definable* if there is a first order formula ϕ in a language $P_1, \ldots, P_n, P_1', \ldots, P_n'$, such that if $\mathfrak{M}$ and $\mathfrak{N}$ are L structures, with the same domain, then the disjoint union of $\mathfrak{M}$ and $\mathfrak{N}$ models ϕ exactly when $\mathfrak{M} \leqslant \mathfrak{N}$.

We can now show that every *first order definable* pre-order is captured by a second order sentence, of the same complexity class as circumscription. They each have only universal second order quantifiers, and are thus Π_1^1, or inductive definitions [24].

Theorem 12. *Let L be a first order language, with predicates $P_1, \ldots, P_n$, and $\leqslant$ a* first order definable *pre-order on structures of L.*

[3] We minimize a pair for simplicity. The extension to an arbitrary finite number is the natural one.

[4] We choose one predicate symbol for simplicity again.

Let $A[P'_1, \ldots, P'_n]$ be a formula of second order logic, without second order quantifiers, with n second order variables P'_i, for $i \leqslant n$.

Then there is a sentence of second order logic Φ with only universal second order quantifiers whose consequences are the formulas entailed in the $\leqslant$ minimal models of $A[P_i/P'_i, P'_j/P_j]$.

Proof. Let ϕ be the witness of the *first order definability* of $\leqslant$, with its $2 * n$ free predicate variables being $P'_i, \ldots, P'_n, P''_1, \ldots, P''_n$.

If $\phi[P_i/P'_i, P'_j/P_j]$ represents the simultaneous replacement of every P_i by P'_i, and P'_j by P_j then,

$$A\left[P_i/P'_i\right] \wedge \forall P'_1, \ldots, P'_n.\left[A \wedge \phi[P_i/P''_i]\right] \to \phi\left[P'_i/P''_i, P_j/P'_j\right]$$

is the required sentence. □

Formulas of the above form has been called *generalized* circumscription by Lifschitz [15]. Here rather than a set of formulas to minimize, we have a formula, with twice as many free predicate variables as the language has predicate constants, that defines the required pre-order.

We now show that circumscription cannot capture all first order axiomatizable pre-orders by considering an example. The following example is more complicated than it need be. However, as it currently is it represents a widely used default, rather than a concocted example.

It is the standard order on structures that captures the default used in *temporal projection*. It uses the situation calculus [23], a formalism for reasoning about change. It is a three sorted language, with situations s, fluents f, and actions a. The predicate $holds(f, s)$ states that the fluent f holds at situation s, while the function $result(a, s)$ gives the situation resulting from doing the action a in situation s. A predicate $ab(a, f, s)$ states that the fluent f changed between s and $result(a, s)$. This order on models has been proposed by Reiter and Lin [19] and Giunchiglia [10] and Costello [3]. It is widely accepted to be the correct solution to the frame problem for problems involving only temporal projection.

Example 2. Consider a pre-order $\leqslant$ on first order structures. Let the language of these structures have three sorts, fluents, actions and situations. The language has two predicates, *holds* which is a predicate on fluent situation pairs, and *ab* which is a predicate on action, fluent, situation triples. It also contains a function from action situation pairs to situations, *result*, and a situation constant $S0$.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be first order structures for this language. We say that $\mathfrak{M} \leqslant \mathfrak{N}$, when if F, A, S are the universes of fluents, actions and situations, in both $\mathfrak{M}$ and $\mathfrak{N}$, and $\mathfrak{M}[P]$ is the interpretation of P in $\mathfrak{M}$ then

$$\forall s \in S, f \in F, a \in A.(\forall f' \in F.\langle f', s\rangle \in \mathfrak{M}[holds] \text{ iff } \langle f', s\rangle \in \mathfrak{N}[holds]) \quad \text{and}$$
$$\langle a, f, s\rangle \in \mathfrak{M}[ab] \text{ implies } \langle a, f, s\rangle \in \mathfrak{N}[ab].$$

We do not relate structures with different universes.

We note that this is clearly a *first order definable* pre-order on models, as the formula

$$(\forall f'.holds(f',s) \equiv holds'(f',s)) \wedge ab(a,f,s) \rightarrow ab'(a,f,s)$$

is the required witness.

This pre-order has an uncountable anti-chain when all the universes, F, A, S are countable. Further, this pre-order has $2^{2^{\aleph_0}}$ upper-sets when all the universes, F, A, S, are countable.

We note that the structures that have a different interpretation of $\lambda f.holds(f, S0)$ are incomparable. Further, every set of structures, for a given triple of universes, which agree on the interpretation of $\lambda f.holds(f, S0)$ form an upper-set. These upper-sets are all clearly disjoint. There are $2^{\aleph_0}$ distinct interpretation of $\lambda f.holds(f, S0)$ and thus there are $2^{2^{\aleph_0}}$ distinct upper-sets.

Proposition 13. *The set of upper-sets of the pre-order generated by a first order circumscription policy, restricted to structures with a given countable universe, has cardinality bounded by* $2^{\aleph_0}$.

Proof. A basis for the upper-sets is given by all instantiations into the minimized formulas. There are at most a countable number of distinct formulas that result from this, as the universe is countable, and the set of formulas is finite, and thus has a finite number of free variables. If the upper-sets of a pre-order have a countable basis, then there are at most $2^{\aleph_0}$ upper-sets, as every upper-set is the result of unions and intersections of upper-sets in the basis. □

Theorem 14. *There are first order definable pre-orders on of first order structures not captured by circumscription.*

Proof. Immediate from Proposition 13 and Example 2. □

This is relevant because the example above that circumscription cannot capture is a natural default that has been used in reasoning about action, a domain where circumscription has been widely applied.

5. Consequence relations

We now compare propositional circumscription to another proposal for characterizing nonmonotonicity. This other proposal, *nonmonotonic consequence relations* [13] has its roots in conditional logic [29]. It differs from conditional logic in two notable ways. It studies only un-nested conditionals, and secondly, it is concerned in ways of completing sets of conditionals. These completions, such as rational closure, do not concern us here.

Definition 17. A *preferential consequence model* M over a finite language $L_{\overline{P}}$ is a triple $\langle W, l, \prec\rangle$, where W is a set of states, $l : W \mapsto 2^{\overline{P}}$ assigns a valuation on $\overline{P}$ to each state, and $\prec$ is a strict partial order on W. The valuation $l(w)$ is called the label of a state w.

This is exactly Definition 5.6 of [13], save that as we consider only finite languages, we do not have a *smoothness* condition.

The preferential consequence relation, corresponding to a *preferential consequence model* $\langle W, l, \prec\rangle$ is a binary relation $\mid\!\sim$ on formulas of $L_{\overline{P}}$, such that $\alpha \mid\!\sim \beta$, when β is true in all the minimal states of the partial order $\prec$ restricted to states w whose labels, $l(w)$ model α. This is exactly Definition 3.13 of [13]. Thus, $\alpha \mid\!\sim \beta$ corresponds to minimal entailment in a preferential consequence model. It is notable that states, which are labeled with models are ordered, rather than models. In order to capture preferential consequence relations Kraus et al. showed that this distinction is necessary, as there are preferential consequence relations all of whose models contain two states that are labeled with the same valuation.

We now show that we can capture all preferential consequence relations using circumscription.

Theorem 15. *Let $\mid\!\sim$ be a preferential consequence relation, over a finite propositional language $L_{\overline{P}}$. There is a finite set of new propositional letters $\overline{P'}$, and a finite set of formulas Γ in $L_{\overline{P}\cup\overline{P'}}$, and a formula $\beta \in L_{\overline{P}\cup\overline{P'}}$, such that for all formulas $\alpha, \phi \in L_{\overline{P}}$,*

$$\alpha \mid\!\sim \phi \quad \textit{if and only if} \quad Circ(\alpha \wedge \beta; \Gamma) \vdash \phi.$$

Proof. We first choose a sufficiently large set of new propositional letters $\overline{P'}$. We then specify our finite set of formulas Γ and our formula β. We then show that the equivalence holds.

Firstly, we use the fact that every preferential consequence relation has a finite model. We can thus choose a finite model $M = \langle W, l, \prec\rangle$. Let the number of states in this model M be n. We let $\overline{P'}$ be a set of $\lceil \log_2 n\rceil$ new propositional letters.

To each state w in W we associate a distinct subset of $\overline{P'}$. Let $l'(w)$ give the set associated with w. There is a set of subsets that are not associated with any state. Let this set of subsets be E.

We now generate a pre-order over a subset of the models of $L_{\overline{P}\cup\overline{P'}}$. We do this by associating a model of $L_{\overline{P}\cup\overline{P'}}$ with every state in W, denoting the model associated with a state w as $L(w)$. The model associated with a state w has the same valuation as the label of the state $l(w)$ on the propositional atoms in $\overline{P}$, and has the valuation $l'(w)$ on the letters in $\overline{P'}$.

No states are associated with the same model, as every distinct state has a distinct valuation over the set $\overline{P'}$.

We now order these models, setting $L(w) \leqslant L(w')$ when $w \prec w'$. We let β be an axiomatization of the range of L, that is, the models that are associated with some state in W.

$\leqslant$ is a partial order on models of β. By Theorem 7 we can find a set of formulas Γ, such that $\leqslant_\Gamma$ restricted to the models that satisfy β is $\leqslant$.

We have specified $\overline{P'}$, β and Γ. We now show that the equivalence

$$\alpha \mid\!\sim \phi \quad \text{if and only if} \quad Circ(\alpha \wedge \beta; \Gamma) \vdash \phi$$

holds for all $\alpha, \phi \in L_{\overline{P}}$.

Let the minimal states whose labels model α be N. The two orders $\prec$ and $\leqslant_\Gamma$ restricted to β models are isomorphic. Also $l(w) \models \alpha$ if and only if $L(w) \models \alpha$, as $L(w)$ and $l(w)$ agree on all letters from $\overline{P}$, and α is in $L_{\overline{P}}$. Therefore, the minimal models of β under $\leqslant_\Gamma$ that model α are the image of N under L. But, ϕ is in $L_{\overline{P}}$, and ϕ is satisfied in the minimal states whose labels model α exactly when ϕ is satisfied in the image of N under L. Therefore, the minimal models of $\alpha \wedge \beta$ under $\leqslant_\Gamma$ model ϕ exactly when ϕ is true in the labels of the minimal states whose labels model α, as required. □

5.1. Discussion

The previous theorem is in strong contrast to Kraus et al. claim that [13, p. 169]:

> The framework of preferential models, therefore, has an expressive power that cannot be captured by negation as failure, circumscription, default logic or autoepistemic logic.

We have shown that circumscription can capture all preferential models over finite propositional languages. The restriction to finite propositional languages is not a severe restriction, as Kraus et al. make the assumption of compactness.

The construction in the proof above essentially uses the introduction of new propositional letters. This is common in circumscription—often new letters called *ab*'s are introduced.

6. Conclusion

We have shown that minimizing a certain formula γ corresponds to stating that, in minimal models semantics, the set of models that satisfy γ is an upper-set of the pre-order on models. We show that all finite pre-orders on models are representable by formula circumscription. We show how the result extends to the infinite case, both in terms of languages and in terms of infinitary connectives and strings of quantifiers.

We have explained what the meaning of minimizing a formula is, in terms of the model theory of circumscription. This greatly clarifies why we minimize certain formulas, and make finding a circumscription policy for a particular pre-order trivial. The correct formulas to minimize are those that correspond to a basis for the family of upper-sets. This paper does not suggest using this connection to implement nonmonotonic reasoning. Rather the paper is an investigation into the theory of a particular form of nonmonotonic reasoning, so that its strengths and weaknesses can be better understood.

We show that *generalized circumscription* as conceived by Lifschitz [15] is strictly stronger in expressive power than formula circumscription, and this extra expressive power is necessary to capture the default of inertia in temporal projection.

Finally we have shown that circumscription is as expressive as preferential consequence relations, in contrast to previous claims.

References

[1] R. Brafman, Statistical first order conditionals, in: L.C. Aiello, J. Doyle, S. Shapiro (Eds.), Principles of Knowledge Representation and Reasoning, Proc. 5th International Conference, Morgan Kaufmann, San Mateo, CA, 1996, pp. 421–431.

[2] T. Costello, Non-monotonicity and change, Ph.D. thesis, Stanford University, 1997.
[3] T. Costello, Change, change, change, three approaches, in: Proc. 15th International Joint Conference on Artificial Intelligence (IJCAI-97), Nagoya, Japan, Morgan Kaufmann, San Mateo, CA, 1997, pp. 1426–1431.
[4] J.P. Delgrande, An approach to default reasoning based on a first-order conditional logic, Artificial Intelligence 36 (1) (1988) 63–90.
[5] J. de Kleer, K. Konolige, Eliminating the fixed predicates from a circumscription, Artificial Intelligence 39 (1989) 391–398.
[6] D.W. Etherington, R.E. Mercer, R. Reiter, On the adequacy of predicate circumscription for closed world reasoning, Computational Intelligence 1 (1985).
[7] D. Gabbay, Theoretical Foundations for Non-Monotonic Reasoning in Expert- Systems, in: K.R. Apt (Ed.), Logics and Models of Concurrent systems, Springer, Berlin, 1985, pp. 439–458.
[8] P. Gärdenfors, Conditionals and changes of belief, Acta Philos. Fennica 30 (1978) 381–404.
[9] P. Gärdenfors, Knowledge in Flux, Cambridge University Press, Cambridge, UK, 1988.
[10] E. Giunchiglia, Determining ramifications in the situation calculus, in: L.C. Aiello, J. Doyle, S. Shapiro (Eds.), Principles of Knowledge Representation and Reasoning: Proc. 5th International Conference (KR-96), Morgan Kaufmann, San Mateo, CA, 1996, pp. 76–86.
[11] P. Gärdenfors, D. Makinson, Revisions of knowledge systems using epistemic entrenchment, in: M.Y. Vardi (Ed.), Proc. 2nd Conference on Theoretical Aspects of Reasoning about Knowledge, Morgan Kaufmann, San Mateo, CA, March 1988, pp. 83–95.
[12] P. Gärdenfors, D. Makinson, Nonmonotonic inference based on expectations, Artificial Intelligence 65 (2) (1994) 197–245.
[13] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, Artificial Intelligence 44 (1) (1990) 167–207.
[14] D.K. Lewis, Counterfactuals, Harvard University Press, Cambridge, MA, 1973.
[15] V. Lifschitz, Some results on circumscription, in: 1st Workshop on Non-monotonic Reasoning, Stanford University, August 1984, pp. 151–164.
[16] V. Lifschitz, Computing Circumscription, in: Proc. 9th International Joint Conference on Artificial Intelligence (IJCAI-85), Los Angeles, CA, 1985, pp. 121–127.
[17] V. Lifschitz, Nested abnormality theories, Artificial Intelligence 74 (1995) 351–365.
[18] D. Lehmann, M. Magidor, Preferential logics: the predicate calculus case, in: R. Parikh (Ed.), Proc. 3rd Conference on Theoretical Aspects of Reasoning about Knowledge, Morgan Kaufmann, San Mateo, CA, March 1990, pp. 57–72,
[19] F. Lin, R. Reiter, State constraints revisited, Journal of Logic and Computation 4 (5) (1994) 655–678.
[20] J. McCarthy, Epistemological problems of artificial intelligence, in: Proc. 5th International Joint Conference on Artificial Intelligence (IJCAI-77), Cambridge, MA, 1977, pp. 223–227.
[21] J. McCarthy, Circumscription—a form of non-monotonic reasoning, Artificial Intelligence 13 (1–2) (1980) 27–39.
[22] J. McCarthy, Applications of circumscription to formalizing commonsense knowledge, Artificial Intelligence 28 (1986) 89–116.
[23] J. McCarthy, P. Hayes, Some philosophical problems from the standpoint of artificial intelligence, in: D. Michie (Ed.), Machine Intelligence 4, Edinburgh University Press, Edinburgh, UK, 1969, pp. 463–502.
[24] Y.N. Moschovakis, Elementary Induction on Abstract Structures, North-Holland, Amsterdam, 1974.
[25] F.P. Ramsey, General propositions and causality (1925), in: D.H. Mellor (Ed.), Foundations: Essays in Philosophy, Logic, Mathematics and Economics, Routledge and K. Paul, London, 1978, pp. 237–257.
[26] P.K. Rathmann, M. Winslett, M. Manasse, Circumscription with homomorphisms: solving the equality and counterexample problem, Journal of the ACM 41 (5) (1994) 819–873.
[27] Y. Shoham, Reasoning about change: time and causation from the standpoint of artificial intelligence, Ph.D. thesis, Yale University, 1986.
[28] R. Stalnaker, A theory of conditionals, in: N. Rescher (Ed.), Studies in Logical Theory, American Philosophical Quarterly Monograph Series, Vol. 2, Blackwell, Oxford, 1968, pp. 98–112. Also in: W. Harper, R.C. Stalnaker, G. Pearce (Eds.), Ifs, Reidel, Dordrecht, 1981.
[29] F. Veltman, Logics for conditionals, Ph.D. thesis, Filosofisch Instituut, Universiteit van Amsterdam, 1986.